3.3 | Derivatives & the Shape of Graphs

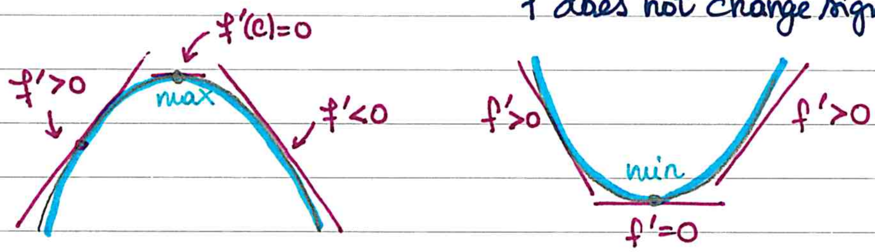
$\boxed{f'}$ → Determines if f is increasing / decreasing:

* If $f'(x) > 0$ on some interval, then f is increasing on that interval.
* If $f'(x) < 0$ on some interval, then f is decreasing on that interval.

The First Derivative Test:

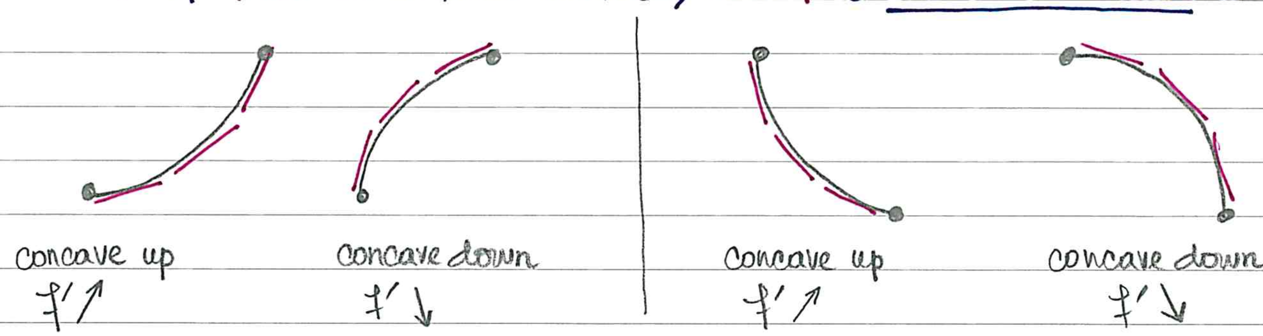
$\boxed{c}$ = critical number of f

- f' changes sign from ⊕ to ⊖ at c => $\boxed{x = c \text{ is a local max}}$
- f' changes sign from ⊖ to ⊕ at c => $\boxed{x = c \text{ is a local min}}$
- f' does not change sign => neither min/max



$\boxed{f''}$ → Determines if f is concave up/down.

* If $f''(x) > 0$ on some interval I, then f is concave upward on I.
* If $f''(x) < 0$ on some interval I, then f is concave downward on I



"concave up" → f lies above its tangent lines.

"concave down" → f lies below its tangent lines.

The Second Derivative Test:

$\boxed{c}$ s.t. $\boxed{f'(c) = 0}$

- $\boxed{f''(c) > 0}$ => local min at c
- $\boxed{f''(c) < 0}$ => local max at c

Inflection Point: A point P on the curve $y = f(x)$ where f is continuous and changes concavity [So: $f''(x) = 0$ and f'' changes sign at x]

3.3. Derivatives & graphs.

1 $f(x) = 8x^3 - 9$

(a) Domain: $(-\infty, \infty)$ or $\mathbb{R}$

(b) Critical no.'s: $f'(x) = 24x^2 \Rightarrow \boxed{0}$ is the only critical no.

x	$-\infty$		0		∞
$f'(x)$	$+$	$+ + +$	0	$+ + +$	
$f(x)$	$-\infty$	$\nearrow$	-9	$\nearrow$	∞

(c) Sign of first derivative?

$f'(x) > 0$ on $(-\infty, 0) \cup (0, \infty)$

$f'(x) < 0$ nowhere

(d) Local min/max? NONE $\rightarrow$ f' does not change sign.

2 $f(x) = 2x + \frac{10}{x}$

(a) Domain: $(-\infty, 0) \cup (0, \infty)$

(b) Critical numbers: $\boxed{f'(x) = 2 - \frac{10}{x^2}}$

$2 - \frac{10}{x^2} = 0 \Rightarrow 2x^2 - 10 = 0 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

(c) Sign of 1st deriv.?

$f'(x) = 2 - \frac{10}{x^2} = \frac{2x^2 - 10}{x^2}$ $\leftarrow$ determines the sign of f' $\leftarrow$ always $\oplus$

x	$-\infty$		$-\sqrt{5}$		0		$\sqrt{5}$		∞
$f'(x)$		$+ +$	0	$-$		$-$	0	$+ +$	
$f(x)$		$\nearrow$	max	$\searrow$		$\searrow$	min	$\nearrow$	

$f'(x) > 0$ on $(-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$; $f'(x) < 0$ on $(-\sqrt{5}, 0) \cup (0, \sqrt{5})$

(d) Local max/min? Local max at $x = -\sqrt{5}$; Local min at $x = \sqrt{5}$

3 $f(x) = \frac{x-8}{x+4}$

(a) Domain: $(-\infty, -4) \cup (-4, \infty)$

(b) C.Pts.: $f'(x) = \frac{(x+4) - (x-8)}{(x+4)^2} = \frac{12}{(x+4)^2} > 0$ always

NONE (-4 is not in the domain)

(c) Sign of 1st deriv.?

$f'(x) > 0$ on $(-\infty, -4) \cup (-4, \infty)$

$f'(x) < 0$ nowhere

(d) No min/max (f' does not change sign).

4 $f(x) = \dfrac{7x^2}{x-2}$

ⓐ Domain: $(-\infty, 2) \cup (2, \infty)$

ⓑ C.Pts.: $f'(x) = \dfrac{14x(x-2) - 7x^2}{(x-2)^2} = \dfrac{14x^2 - 28x - 7x^2}{(x-2)^2} = \dfrac{7x^2 - 28x}{(x-2)^2}$

$f'(x) = \dfrac{7x(x-4)}{(x-2)^2} \Rightarrow$ C.Pts.: $x = \boxed{0, 4}$

ⓒ Sign of 1^{st} deriv.?

x	$-\infty$		0		2		4		∞
$f'(x)$	$+$	$+$	0	$-$ |	$-$		0	$+$	$+$
$f(x)$	$\nearrow$		max	$\searrow$	$\searrow$		min	$\nearrow$	

$f'(x) > 0$ on $(-\infty, 0) \cup (4, \infty)$
$f'(x) < 0$ on $(0, 2) \cup (2, 4)$

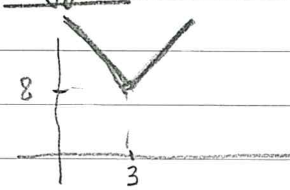
ⓓ Local min/max? Local max at $x = 0$; Local min at $x = 4$.

5 $f(x) = |x-3| + 8 = \begin{cases} -(x-3)+8, & \text{if } x < 3 \\ +(x-3)+8, & \text{if } x \geq 3 \end{cases} = \begin{cases} -x+11, & x < 3 \\ x+5, & x \geq 3 \end{cases}$

ⓐ Domain: $(-\infty, \infty)$

ⓑ C.Pts.:

$f'(x) = \begin{cases} -1, & x < 3 \\ +1, & x > 3 \end{cases}$ Not diff'ble at $x=3$! $\Rightarrow$ $x=3$ C.Pt.



ⓒ $f'(x) > 0$ on $(3, \infty)$, $f'(x) < 0$ on $(-\infty, 3)$

ⓓ Local min at $x = 3$, no local max

6 $f(x) = 9(x-7)^{2/3} + 3$ ⓐ Domain: $(-\infty, \infty)$

ⓑ C.Pts.: $f'(x) = 9 \cdot \frac{2}{3}(x-7)^{-1/3} = \dfrac{6}{\sqrt[3]{x-7}}$ f' dne at $x=7$ $\leftarrow$ C.Pt.

ⓒ Sign of 1^{st} deriv.:

x	$-\infty$			7			∞
$f'(x)$	$-$	$-$	$-$	|	$+$	$+$	$+$
$f(x)$		$\searrow$		3 min		$\nearrow$	

$f'(x) < 0$ on $(-\infty, 7)$
$f'(x) > 0$ on $(7, \infty)$

ⓓ Local min/max? Local min at $x=7$ (no local max)

7 $f(x) = 2\sqrt{x} - 3x$. Where is f increasing / decreasing?

$$f'(x) = \frac{1}{\sqrt{x}} - 3 = \frac{1 - 3\sqrt{x}}{\sqrt{x}}$$

$1 - 3\sqrt{x} = 0$

$1 = 3\sqrt{x}$

$1/3 = \sqrt{x}$

$1/9 = x$

x	$-\infty$		0		$1/9$		∞
$f'(x)$				$+$	0	$- - -$	
$f(x)$				$\nearrow$	max	$\searrow$	

$f \uparrow$ on $(0, 1/9)$.

$f \downarrow$ on $(1/9, \infty)$.

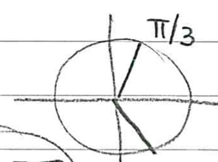
8 $$f(x) = \frac{-1}{|x-7|} = \begin{cases} \frac{-1}{-(x-7)}, & \text{if } x < 7 \\ \frac{-1}{+(x-7)}, & \text{if } x > 7 \end{cases} = \begin{cases} \frac{1}{x-7}, & \text{if } x < 7 \\ \frac{-1}{x-7}, & \text{if } x > 7 \end{cases}$$

$$f'(x) = \begin{cases} \frac{-1}{(x-7)^2}, & \text{if } x < 7 \\ \frac{+1}{(x-7)^2}, & \text{if } x > 7 \end{cases}$$

$\Rightarrow f'(x) < 0$ on $(-\infty, 7)$, $f \downarrow$

$\Rightarrow f'(x) > 0$ on $(7, \infty)$, $f \uparrow$

9 $f(x) = 2\sin(x) - x$ on $[0, 2\pi]$



$f'(x) = 2\cos(x) - 1$

$f'(x) = 0,\ x \in [0, 2\pi] \Rightarrow \cos(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$

x	0		$\frac{\pi}{6}$		$\frac{\pi}{3}$		π		$\frac{5\pi}{3}$			2π
$f'(x)$	1	$\oplus$	$\sqrt{3}-1$	$\oplus$	0	$\ominus$	-3	$\ominus$	0	$\oplus$	$\oplus$	1
$f(x)$		$\nearrow$			max	$\searrow$			min	$\nearrow$		

$f \uparrow$ on $(0, \pi/3) \cup (5\pi/3, 2\pi)$

$f \downarrow$ on $(\pi/3, 5\pi/3)$

10 $f(x) = 2x^3 - 6x + 3$ Convexity?

$f'(x) = 6x^2 - 6$

$f''(x) = 12x$

Inflection pt.

x		0	
$f''(x)$	$- -$	0	$+ + +$
$f(x)$	$\cap$		$\cup$

$f \frown$ on $(-\infty, 0)$

$f \smile$ on $(0, \infty)$

11 $f(x) = x^3 - 9x^2 + 2x + 11$ Domain: $(-\infty, \infty)$

ⓐ. Critical no.'s?

$f'(x) = 3x^2 - 18x + 2$

$f'(x) = 0 \Rightarrow \Delta = 18^2 - 4 \cdot 2 \cdot 3 = 300 \Rightarrow x_{1,2} = \frac{18 \pm \sqrt{300}}{6} = \frac{18 \pm 10\sqrt{3}}{6}$

$= \frac{9 \pm 5\sqrt{3}}{3}$

ⓑ Local min/max? by 2^{nd} Deriv. Test.

$f''(x) = 6x - 18$

$f''\left(\frac{9-5\sqrt{3}}{3}\right) = \cancel{18} - 10\sqrt{3} - \cancel{18} = -10\sqrt{3} < 0 \Rightarrow$ local max

$f''\left(\frac{9+5\sqrt{3}}{3}\right) = \cancel{18} + 10\sqrt{3} - \cancel{18} = 10\sqrt{3} > 0 \Rightarrow$ local min

12 $f(x) = 4x - 5x^{4/5}$ Domain: $(-\infty, \infty)$.

ⓐ. Critical no.'s:

$f'(x) = 4 - 5 \cdot \frac{4}{5} x^{-1/5} = 4 - \frac{4}{\sqrt[5]{x}} = \frac{4(\sqrt[5]{x} - 1)}{\sqrt[5]{x}}$

$f'(x) = 0$ at $x = 1$ and $f'(x)$ dne at $x = 0$ $\Rightarrow$ C. no.'s: $x = 0, 1$

ⓑ Local min/max by 2^{nd} Deriv. Test.

$f''(x) = -4 \cdot \frac{-1}{5} x^{-6/5} = \frac{4}{5(\sqrt[5]{x})^6}$

$f''(0)$ dne $\Rightarrow$ no conclusion; $f''(1) = \frac{4}{5} > 0$ $\Rightarrow$ local min

13 Derivative of f is: $f'(x) = x^3 + 2x^2 - 3x$

ⓐ Critical no.'s? $f'(x) = 0 \Rightarrow x(x^2 + 2x - 3) = 0$

$x(x+3)(x-1) = 0 \Rightarrow x = 0, 1, -3$

ⓑ Classify by 2^{nd} Deriv. Test:

$f''(x) = 3x^2 + 4x - 3$

$f''(0) = -3 < 0 \Rightarrow$ local max

$f''(1) = 4 > 0 \Rightarrow$ local min

$f''(-3) = 27 - 12 - 3 = 12 > 0 \Rightarrow$ local min